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KRISTER SEGERBERG

Results in non-classical propositional logic

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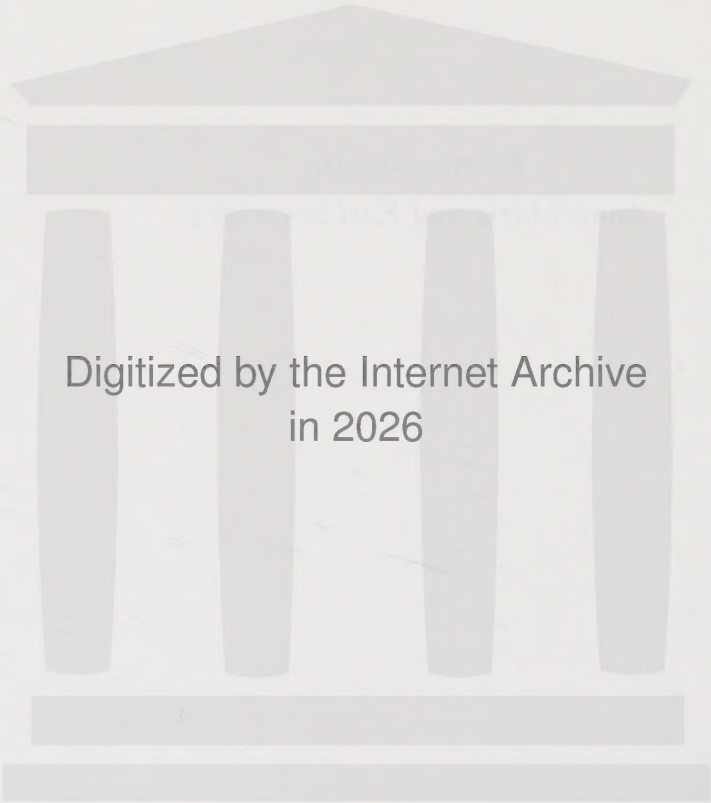
To my parents,
Anne-Marie and Karl Segerberg



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PREFACE

These notes have been compiled in order to comply with Swedish university regulations which allow published papers to be used as a doctoral dissertation if they are accompanied by a printed summary. The five papers making up this dissertation are those listed under the author's name in the references. They were written during the author's sojourn at Stanford University 1965—68, and came into being as follows. The first inspiration for **[1]** was derived from listening to Georg Henrik von Wright's stimulating 1964 lectures on tense-logic at the Summer University at Vasa, Finland. The author's interest in Arthur Prior's Q, which in time led to the writing of **[2]**, was aroused by a talk of Bruce White in Dagfinn Føllesdal's seminar in the spring term of 1966. Already at the end of 1965 the author had had the opportunity to read a copy of what is referred to in **[2]** as an unpublished paper of Dana Scott, where Scott presents his completeness and decidability proofs for the fundamental classical modal logics. The idea of applying his approach to Prior's maverick logic was obvious. A draft of **[2]** was discussed in Stig Kanger's seminar in the summer term 1966. Lennart Åqvist read **[1]** and **[2]** in manuscript, and offered helpful criticism. **[3]** and **[4]** were suggested by E. J. Lemmon's admirable discussion in Lemmon & Scott [1]. Work on **[5]** was begun in the winter term of 1967 when, having just read Kripke [4], the author wrote a term paper for Georg Kreisel's course on intuitionistic logic. Dag Prawitz read part of the manuscript of **[5]** and suggested a number of improvements.

Although I am grateful to all of the persons mentioned above, it is to Dana Scott I owe the greatest debt. In countless conversations he has patiently explained his elegant ideas about modal logic; ideas I am making use of in this dissertation, although some

of the elegance may have worn off in the transition. To be his student has been incomparably stimulating and rewarding. Indeed, my whole time in Stanford's philosophy department, under Patrick Suppes, has been an extremely valuable experience, intellectually and otherwise.

It is a particular pleasure to be able to present this dissertation while Konrad Marc-Wogau is still holding the chair of theoretical philosophy at Uppsala. The years when I had the privilege of studying under him, 1959—65, have no doubt indelibly influenced my thinking on philosophical matters. For his sake I wish I could have written a more philosophical treatise. That my interests led me into formal logic may have disappointed him—so much more do I appreciate his never ceasing support and loyalty. I should also like to thank Sören Halldén for his encouragement during the last few years, and for his kindness in providing space in *Theoria* for the expeditious publication of my papers. Without him this dissertation would have been presented much later.

My friends Brian F. Chellas and David Miller have given invaluable assistance in connection with the problems that are bound to confront anyone who sets out to write in a language not his own.

On the financial side I gratefully acknowledge support from the following sources: a 'doktorandstipendium' from Uppsala University, 1966—68; a Fulbright Travel Grant from the U.S. Educational Commission in Sweden; a fellowship from Ephraim and Amelia Weiss's fund, Stanford University, 1966—67; a Stanford--Wilson Dissertation Fellowship, 1967—68; and a Per Erik Lindahl Grant, awarded by the Royal Academy of Science, Stockholm, 1967—68.

I will not attempt to express my gratitude and indebtedness to my wife for the many years of our lives that have gone into my education. This dissertation would be dedicated to her, were it not that by dedicating it to my parents, I am able to fulfil at last another undefinable obligation of even longer standing.

Stanford, California
February 1968

K. S.

INTRODUCTION

This booklet is a summary of results that the author has obtained in three related domains of propositional logic: classical modal logic, three-valued modal logic, and the theory of extensions of the so-called minimal logic. With the exception of **[1]**, the author's work essentially consists in the application to new material of methods worked out by others. Among publications, the chief source of inspiration has been the outstanding exposition Lemmon & Scott [1]. Little effort is made here to trace the history of the ideas on which our work is based. Undoubtedly it is Kripke's writings that have most signally influenced modern modal logic, but a complete list of our creditors would also have to include, besides Lemmon and Scott, at least Henkin, Hintikka, Kanger, McKinsey, and Tarski.

In this introduction we bring together a number of definitions needed in the sequel. Chapter I presents a résumé of results due to others. In the last three chapters the author's own contributions in **[1]**, **[2]**, **[3]**, **[4]**, **[5]** are summarized. The terminology used here differs in places from that used in the papers.

The alphabets used in this work will consist of (i) denumerably many propositional letters $P_0, P_1, \ldots, P_k, \ldots$; (ii) a finite number of (primitive) logical constants, to each of which is associated a non-negative integer, called its degree; (iii) the brackets [and]. Let an alphabet be given whose logical constants are $o_1, \ldots, o_n$, with degrees $d_1, \ldots, d_n$, respectively. Then the set $\Phi(o_1, \ldots, o_n)$ of all formulas generated by $o_1, \ldots, o_n$ is defined as the smallest set Σ such that (i) every propositional letter is in Σ; and (ii) for $1 \leq i \leq n$, if $A_1, \ldots, A_{d_i}$ are in Σ, then so is $o_i[A_1, \ldots, A_{d_i}]$. The following primitive logical constants will be used:

Chapters I and II: $\bot$, $\rightarrow$, $\Box$, with degrees 0, 2, 1.
Chapter III: $\bot$, $\rightarrow$, T, $\boxed{1}$, $\boxed{2}$, with degrees 0, 2, 1, 1, 1.
Chapter IV: $\bot$, $\wedge$, $\vee$, $\rightarrow$, with degrees 0, 2, 2, 2.

Whenever **o** is a logical constant of degree 2, we write [**A o B**] instead of **o**[**A**, **B**]. We use $\neg$**A** throughout to abbreviate [**A** $\rightarrow \bot$]. Where $\wedge$ and $\vee$ are not primitive they are introduced as abbreviations in the usual way. **A** $\leftrightarrow$ **B** abbreviates [**A** $\rightarrow$ **B**] $\wedge$ [**B** $\rightarrow$ **A**]. In Chapter II, $\Diamond$**A** abbreviates $\neg\Box\neg$**A**. In Chapter III, $+$**A** abbreviates [$\neg$T$\neg$**A** $\rightarrow$ T**A**]. Brackets are dropped in accordance with the usual canons; the main convention is that constants of degree 1 bind more strongly than constants of degree 2, and that among the latter $\wedge$ and $\vee$ bind more strongly than $\rightarrow$ and $\leftrightarrow$. Bold letters from the beginning of the Latin alphabet will be used as variables ranging over the set of formulas under consideration. The symbols $\sim$, &, v, $\Rightarrow$, $\Leftrightarrow$, $\forall$, $\exists$ are occasionally used as a shorthand notation in the meta-language; conventions for dropping parentheses parallel those for dropping brackets. Inclusion is denoted by $\subseteq$, proper inclusion by $\subset$.

Let Σ be a set of formulas. Σ is said to be closed under modus ponens iff, for all **A** and **B**, if **A** $\rightarrow$ **B** $\in \Sigma$ and **A** $\in \Sigma$ then **B** $\in \Sigma$. Σ is closed under subformulas iff, for all **A** and **B**, if **A** is a subformula of **B** and **B** $\in \Sigma$, then **A** $\in \Sigma$. Σ is closed under modalities iff, for all **A**, if **A** $\in \Sigma$ then $\neg$**A** $\in \Sigma$ and $\Box$**A** $\in \Sigma$.

Let Φ be the set of formulas generated by a given set of logical constants. In general, by a logic (over Φ) we understand a set L that is closed under modus ponens; however, in Chapter III we only require closure under a certain weakened form of modus ponens. By PC (over Φ) we mean the set of formulas in Φ that are two-valued tautologies. If $\Sigma \subseteq \Phi$, a formula **A** is said to be derivable from Σ in L, or L-derivable from Σ, in symbols $\Sigma \vdash_L$ **A**, iff, for some $n \geq 0$, there are formulas $\mathbf{B}_1, \ldots, \mathbf{B}_n \in \Sigma$ such that $\mathbf{B}_1 \wedge \ldots \wedge \mathbf{B}_n \rightarrow \mathbf{A} \in L$; if $n = 0$ this means that **A** $\in L$. (Conjunction is associative in all logics considered here, so we may safely dispense with the brackets.) Σ is L-inconsistent iff, for all **A**, **A** is derivable from Σ in L; if at least some **A** is not so derivable, Σ is L-consistent. If Σ is the smallest set of formulas such that

(i) Σ contains the logic L, and (ii) all instances of the schemata $S_1, \ldots, S_n$ are in Σ, then Σ is said to be the logic defined by L and $S_1, \ldots, S_n$. If Σ is the smallest logic that satisfies (i)—(ii) as well as the condition that (iii) Σ is normal—concepts of normality are defined later—then Σ is said to be the normal logic defined by L and $S_1, \ldots, S_n$.

We shall later introduce semantical notions, among them validity in a class of frames. The definition of frame will vary from chapter to chapter, but the following concepts are invariant. We write $\models_{\mathsf{C}} \mathbf{A}$ if C is a class of frames and $\mathbf{A}$ is valid in C. Let L be a logic. Then L is consistent with respect to C iff $\forall \mathbf{A} \in \Phi(\vdash_L \mathbf{A} \Rightarrow \Rightarrow \models_{\mathsf{C}} \mathbf{A})$. L is complete with respect to C iff $\forall \mathbf{A} \in \Phi(\models_{\mathsf{C}} \mathbf{A} \Rightarrow \vdash_L \mathbf{A})$. L is determined by C iff L is both consistent and complete with respect to C.

I. PRELIMINARIES

In this chapter and the next, let $\Phi = \Phi(\bot, \rightarrow, \Box)$. A (*classical*) *modal logic* is a logic over Φ which contains *PC*. By *K* we understand the smallest modal logic *L* such that *L* contains all instances of the schema

$$\#1.\ \Box[\mathbf{A} \rightarrow \mathbf{B}] \rightarrow [\Box\mathbf{A} \rightarrow \Box\mathbf{B}]$$

and is closed under the necessitation rule,

$$\text{RN.}\ \frac{\mathbf{A}}{\Box\mathbf{A}}$$

(that is, if $\mathbf{A} \in L$ then $\Box\mathbf{A} \in L$). A modal logic that contains *K* and is closed under RN is said to be *normal*. Consider the following schemata:

$$\#2.\ \Box\mathbf{A} \rightarrow \mathbf{A}.$$
$$\#3.\ \Diamond\Box\mathbf{A} \rightarrow \mathbf{A}.$$
$$\#4.\ \Box\mathbf{A} \rightarrow \Box\Box\mathbf{A}.$$

The Feys-von Wright system *T* is the normal logic defined by *K* and #2; the Brouwer system, that defined by *K* and ##2,3; and Lewis's *S*4 and *S*5, those defined by *K* and ##2,4 and by *K* and ##2, 3, 4.

We now turn to semantics. $\mathfrak{F} = \langle X, R\rangle$ is said to be a frame if *X* is a non-empty set (the domain of $\mathfrak{F}$) and *R* is a subset of $X \times X$ (the accessibility relation of $\mathfrak{F}$). A valuation in $\mathfrak{F}$ is a function from the set of propositional letters into the power set of *X*. If $\mathfrak{F} = \langle X, R\rangle$ is a frame, and φ is a valuation in $\mathfrak{F}$, then $\mathfrak{A} = \langle X, R, \varphi\rangle$ is a model, namely the model defined on $\mathfrak{F}$ by φ; and conversely, $\mathfrak{F}$ is said to be the frame of $\mathfrak{A}$, and φ the valuation of $\mathfrak{A}$. Subframes and submodels are defined in the obvious manner. In order to ease the exposition we allow certain liberties of terminology, not literally accurate, but hopefully suggestive enough to bar confu-

sion. To give one example, we say that x belongs to a transitive model $\mathfrak{A}$, when we ought to say that x belongs to the domain of $\mathfrak{A}$ and that the accessibility relation of $\mathfrak{A}$ is transitive.

If $\mathfrak{A} = \langle X, R, \varphi \rangle$ is a model, and x is an element in $\mathfrak{A}$, then truth of a formula at x in $\mathfrak{A}$ is defined as follows. Let T and F be any two fixed objects and read '**A** is true, or holds, at x in $\mathfrak{A}$' for $|\!|\mathbf{A}|\!|_x^{\mathfrak{A}} = \mathrm{T}$, and '**A** is false, or fails, at x in $\mathfrak{A}$' for $|\!|\mathbf{A}|\!|_x^{\mathfrak{A}} = \mathrm{F}$.

1. *For all* $k \geq 0$, $|\!|\mathbf{P}_k|\!|_x^{\mathfrak{A}} = \begin{cases} \mathrm{T}, \textit{ if } x \in \varphi(\mathbf{P}_k), \\ \mathrm{F}, \textit{ if } x \notin \varphi(\mathbf{P}_k). \end{cases}$
2. $|\!|\bot|\!|_x^{\mathfrak{A}} = \mathrm{F}$.
3. $|\!|\mathbf{A} \to \mathbf{B}|\!|_x^{\mathfrak{A}} = \begin{cases} \mathrm{T}, \textit{ if } |\!|\mathbf{A}|\!|_x^{\mathfrak{A}} = \mathrm{F} \textit{ or } |\!|\mathbf{B}|\!|_x^{\mathfrak{A}} = \mathrm{T}, \\ \mathrm{F}, \textit{ if } |\!|\mathbf{A}|\!|_x^{\mathfrak{A}} = \mathrm{T} \textit{ and } |\!|\mathbf{B}|\!|_x^{\mathfrak{A}} = \mathrm{F}. \end{cases}$
4. $|\!|\Box\mathbf{A}|\!|_x^{\mathfrak{A}} = \begin{cases} \mathrm{T}, \textit{ if, for all } y \textit{ such that } xRy, |\!|\mathbf{A}|\!|_y^{\mathfrak{A}} = \mathrm{T}, \\ \mathrm{F}, \textit{ if, for some } y \textit{ such that } xRy, |\!|\mathbf{A}|\!|_y^{\mathfrak{A}} = \mathrm{F}. \end{cases}$

A is said to be true, or to hold, in $\mathfrak{A}$ iff **A** is true at all x in $\mathfrak{A}$. **A** is said to be false, or to fail, in $\mathfrak{A}$ iff **A** is not true in $\mathfrak{A}$. **A** is valid in a frame $\mathfrak{F}$ iff **A** is true in all models on $\mathfrak{F}$. **A** is valid in a class C of frames iff **A** is valid in all frames in C.

Let L be a classical modal logic. A set $\Sigma \subseteq \Phi$ is said to be L-maximal iff

(i) $L \subseteq \Sigma$;
(ii) Σ is L-consistent;
(iii) for all **A** and **B**, $\mathbf{A} \to \mathbf{B} \in \Sigma$ iff $\mathbf{A} \notin \Sigma$ or $\mathbf{B} \in \Sigma$.

Note that if Σ is L-maximal, then $\mathbf{A} \in \Sigma$ iff $\neg\mathbf{A} \notin \Sigma$. Hence Σ is maximal in the sense that $\Sigma \cup \{\mathbf{A}\}$ is L-inconsistent if $\mathbf{A} \notin \Sigma$. The *canonical model* (for L) is the model $\mathfrak{M}_L = \langle X_L, R_L, \varphi_L \rangle$, where

$X_L = \{x\colon x \subseteq \Phi \ \&\ x \textit{ is } L\textit{-maximal}\}$;
$R_L = \{\langle x, y\rangle\colon x, y \in X_L \ \&\ \forall\mathbf{A}\{\Box\mathbf{A} \in x \Rightarrow \mathbf{A} \in y\}\}$;
$\varphi_L(\mathbf{P}_k) = \{x\colon x \in X_L \ \&\ \mathbf{P}_k \in x\}$, for all $k \geq 0$.

The fundamental result is this:

LEMMA 1.1. (Scott) *If L is a normal classical modal logic, then for all x in $\mathfrak{M}_L$ and for all formulas* **A**, $|\!|\mathbf{A}|\!|_x^{\mathfrak{M}_L} = \mathrm{T}$ *iff* $\mathbf{A} \in x$.

From this lemma it easily follows that K is complete with respect to the class of all frames. For suppose that **A** is not K-derivable.

Then $\{\neg \mathbf{A}\}$ can be embedded in a K-maximal set x. Since $\mathbf{A} \notin x$, the lemma entails that $\mathbf{A}$ is false at x in the canonical model for K. So there is a frame in which $\mathbf{A}$ is not valid. Completeness results for most classical modal logics in the literature are also easily derivable. To take one example, $\mathfrak{M}_{S4}$ is readily seen to be reflexive and transitive; hence $S4$ is complete with respect to the class of all reflexive and transitive frames. (For the earliest results of this kind, see Hintikka [1], [2], Kanger [2], [3], Kripke [1], [2], and Montague [1]. Makinson [1] contains a Henkin-style proof, essentially the same as Scott's, of a result very close to Lemma 1.1. Cf. Kaplan [1].)

Let $\mathfrak{A} = \langle X, R, \varphi \rangle$ be any model, and x any element in $\mathfrak{A}$. Then by the model generated from $\mathfrak{A}$ by x we mean the model $\mathfrak{A}^x = \langle X^x, R^x, \varphi^x \rangle$, where

$$X^x = \{y \in X: xR^*y\};$$
$$R^x = R \cap (X^x \times X^x);$$
$$\varphi^x(\mathbf{P}_k) = \varphi(\mathbf{P}_k) \cap X^x, \text{ for all } k \geq 0.$$

(R^* is the ancestral of R; that is to say, the least reflexive relation S such that, for all x, y, z, if xSy and yRz then xSz.)

LEMMA 1.2. (Lemmon, Scott) *For all models* $\mathfrak{A}$, *elements* x *in* $\mathfrak{A}$, *and formulas* $\mathbf{A}$, *if* y *is in* $\mathfrak{A}^x$ *then* $\|\mathbf{A}\|_y^{\mathfrak{A}} = \|\mathbf{A}\|_y^{\mathfrak{A}^x}$.

Let $\mathfrak{A} = \langle X, R, \varphi \rangle$ be any model, and let Ψ be any subset of Φ closed under subformulas. Ψ induces the following equivalence relation on X:

$$x \equiv_\Psi y \text{ iff } \forall \mathbf{A} \in \Psi (\|\mathbf{A}\|_x^{\mathfrak{A}} = \|\mathbf{A}\|_y^{\mathfrak{A}}).$$

The equivalence class of x under $\equiv_\Psi$ is denoted by $[x]_\Psi$, or $[x]$ for short. $\mathfrak{A}' = \langle X', R', \varphi' \rangle$ is said to be a filtration of $\mathfrak{A}$ through Ψ if

(i) $X' = \{[x] : x \in X\}$;
(ii) for all x and y, $xRy \Rightarrow [x]R'[y]$;
(iii) for all x and y, $[x]R'[y] \Rightarrow \forall \mathbf{A}(\Box \mathbf{A} \in \Psi \;\&\; \|\Box \mathbf{A}\|_x^{\mathfrak{A}} = \mathrm{T} \Rightarrow \Rightarrow \|\mathbf{A}\|_y^{\mathfrak{A}} = \mathrm{T})$;
(iv) $\varphi'(\mathbf{P}_k) = \{[x]: x \in X \;\&\; \mathbf{P}_k \in \Psi \;\&\; x \in \varphi(\mathbf{P}_k)\}$, for all $k \geq 0$.

Filtrations are obviously models. The filtration whose accessibility relation is contained in every relation satisfying (ii) and (iii) is called the smallest filtration.

LEMMA 1.3. (McKinsey, Lemmon, Scott) *If* Ψ *is a set of formulas closed under subformulas, and if* $\mathfrak{B}$ *is a filtration of* $\mathfrak{A}$ *through* Ψ, *then for all* x *in* $\mathfrak{A}$ *and for all* A *in* Ψ, $\mathrm{A}\|_x^{\mathfrak{A}} = \|\mathrm{A}\|_{[x]}^{\mathfrak{B}}{}_{\Psi}$.

The author's contribution to this lemma consists in the minor observation that closure under subformulas is a sufficiently strong condition on Ψ to make the lemma true. Previous authors restricted attention to the case when Ψ is the set of subformulas of a given formula. This is indeed the important case, but there are applications of the lemma which do not seem to be possible except in its general form; see the proof of Lemma 2.3 below.

II. RESULTS IN CLASSICAL MODAL LOGIC

Let $\mathfrak{A} = \langle X, R, \varphi \rangle$ and $\mathfrak{A}' = \langle X', R', \varphi' \rangle$ be models. Suppose f is a function from X to X' such that

(i) f is onto;
(ii) for all x and y, $xRy \Rightarrow fxR'fy$;
(iii) for all x and y, $fxR'fy \Rightarrow \exists z(fz = fy \ \& \ xRz)$;
(iv) for all $k \geq 0$, $f[\varphi(\mathbf{P}_k)] = \varphi'(\mathbf{P}_k)$.

Then, for lack of a better name, f is said to be a pseudo-epimorphism from $\mathfrak{A}$ to $\mathfrak{A}'$.

LEMMA 2.1. *Let f be a pseudo-epimorphism from a model $\mathfrak{A}$ to a model $\mathfrak{A}'$. Then for all x in $\mathfrak{A}$ and all formulas* **A**, $\mathbf{A}\,{}^{\mathfrak{A}}_{x} = \mathbf{A}\,{}^{\mathfrak{A}'}_{fx}$; *hence,* **A** *is true in* $\mathfrak{A}$ *iff* **A** *is true in* $\mathfrak{A}'$.

Another completeness result for *S*4, in addition to the one mentioned in Chapter I, is the following:

THEOREM 2.2. *S*4 *is determined by the class of all partially ordered frames.*

PROOF. It is enough to show that if **A** is not derivable in *S*4, then **A** fails in some partially ordered frame. It is known (and indeed provable by means of Lemmata 1.1 and 1.3) that if **A** is not *S*4-derivable then there exists a finite reflexive transitive model in which **A** fails. Hence, by Lemma 2.1, it is enough to show that, given such a model $\mathfrak{A} = \langle X, R, \varphi \rangle$, we can construct a partially ordered model $\mathfrak{B} = \langle Y, S, \psi \rangle$ and a pseudo-epimorphism f from $\mathfrak{B}$ to $\mathfrak{A}$.

We say that a set C is an *R-cluster* if (i) C contains at least two elements; (ii) for all x and y in C, xRy and yRx; and (iii) there is no superset of C of which (ii) is true. Thus, if R is not anti-sym-

metric, there must exist R-clusters in $\mathfrak{A}$. Since $\mathfrak{A}$ is finite, it has only finitely many R-clusters, say $C_1, \ldots, C_m$, where each cluster $C_i = \{z_{i1}, \ldots, z_{ir_i}\}$, for some $r_i \geq 2$. Let $Z = \{z_{ij}^n : 1 \leq i \leq m \ \& \ 1 \leq j \leq r_i \ \& \ 1 \leq n\}$ be any set of distinct elements not in X. Define

$$Y = (X - \bigcup_{i=1}^{m} C_i) \cup Z;$$

$$xSy \Leftrightarrow \begin{cases} x, y \notin Z \ \& \ xRy; \text{ or } x = z_{ij}^n \ \& \ y \notin Z \ \& \ z_{ij}Ry; \\ \text{or } x \notin Z \ \& \ y = z_{ij}^n \ \& \ xRz_{ij}; \text{ or } x = z_{ij}^n \ \& \\ y = z_{i'j'}^{n'} \ \& \ (i \neq i' \ \& \ z_{ij}Rz_{i'j'}, \text{ or } i = i' \ \& \\ n < n', \text{ or } i = i' \ \& \ n = n' \ \& \ j \leq j'); \end{cases}$$

$$\psi(\mathbf{P}_k) = (\varphi(\mathbf{P}_k) \cap Y) \cup \{z_{ij}^n : n \geq 1 \ \& \ z_{ij} \in \varphi(\mathbf{P}_k)\}, \text{ for all } k \geq 0.$$

$\mathfrak{B} = \langle Y, S, \psi \rangle$ is partially ordered; and the following defines a pseudo-epimorphism f from $\mathfrak{B}$ to $\mathfrak{A}$:

$$fx = \begin{cases} x, \text{ if } x \in X, \\ z_{ij}, \text{ if, for some } n \geq 1, \ x = z_{ij}^n. \end{cases}$$

Q.E.D.

*S*4.1, first defined in McKinsey [1], is the normal logic defined by *S*4 and the schema

$$\#5. \ \Box \Diamond \mathbf{A} \rightarrow \Diamond \Box \mathbf{A}.$$

We shall next give completeness results for *S*4.1 and two of its extensions. We say that a binary relation R has the McKinsey property (on a set X) if $\forall x \in X \exists y \in X(xRy \ \& \ \forall z,u \in X(yRz \ \& \ yRu \Rightarrow z = u))$. A McKinsey model (frame) is one whose accessibility relation possesses the McKinsey property.

LEMMA 2.3. *Let L be a normal logic at least as strong as S*4.1. *Let Ψ_0 be a finite subset of Φ and suppose Ψ is the closure of Ψ_0 under subformulas and modalities. Let $\mathfrak{M}$ be the canonical model $\mathfrak{M}_L$, or a model generated from $\mathfrak{M}_L$. Then there exists a filtration of $\mathfrak{M}$ through Ψ which is reflexive and transitive, and has the McKinsey property.*

PROOF. Let $\mathfrak{A} = \langle X, R, \varphi \rangle$ be the smallest filtration of $\mathfrak{M}$ through Ψ. It is easily seen that $\mathfrak{A}$ will be reflexive and transitive. Since there are only finitely many irreducible modalities in *S*4.1, X will

be finite, for even though Ψ will contain infinitely many elements, only a finite number of them are logically distinct. Therefore, to show that $\mathfrak{A}$ is a McKinsey model, it will be sufficient to prove that it contains no final clusters, that is to say, to prove that for every S-cluster C in $\mathfrak{A}$ there exists an element $\zeta \in C$ such that if $\xi \in C$ then $\xi S \zeta$. Proceeding by reductio ad absurdum, assume that C is an S-cluster for which no such ζ exists. Let ξ and η be any two distinct elements in C. Then there is some $A \in \Psi$ such that $A \overset{\mathfrak{A}}{\underset{\xi}{}} = T$ and $A \overset{\mathfrak{A}}{\underset{\eta}{}} = F$. Hence, for all $\theta \in C$, $\Diamond A \overset{\mathfrak{A}}{\underset{\theta}{}} = T$. Since by hypothesis there is no $\zeta \in C$ such that $\xi S \zeta$, it follows that $\Box \Diamond A \overset{\mathfrak{A}}{\underset{\xi}{}} = T$. Pick any $\chi \in \xi$. Then χ is an element in $\mathfrak{M}$ and by Lemmata 1.1, 1.2, and 1.3, $\Box \Diamond A \in \chi$. Since L contains $S4.1$, $\Diamond \Box A \in \chi$, so by the same lemmata $\Diamond \Box A \overset{\mathfrak{A}}{\underset{\chi}{}} = T$. Hence there exists some λ such that $\xi S \lambda$ and $\Box A \overset{\mathfrak{A}}{\underset{\lambda}{}} = T$. Our initial assumption, plus the fact that ξ belongs to an S-cluster, implies that $\lambda \in C$. Therefore $\lambda S \eta$, whence $A \overset{\mathfrak{A}}{\underset{\eta}{}} = T$. Thus the desired contradiction has been attained. Q.E.D.

$S4.2$ is the normal logic defined by $S4$ and the schema

$$\ast 6.\quad \Diamond \Box \mathbf{A} \rightarrow \Box \Diamond \mathbf{A}.$$

$S4.3$ is the normal logic defined by $S4$ and the schema

$$\ast 7.\quad \Box[\Box \mathbf{A} \rightarrow \mathbf{B}] \vee \Box[\Box \mathbf{B} \rightarrow \mathbf{A}].$$

It was proved in Dummett & Lemmon [1] that $S4 \subset S4.2 \subset S4.3$, and that $S4.2$ is determined by the class of all directed frames, $S4.3$ by the class of all linear frames. (A binary relation R is directed (on a set X) iff R is reflexive and transitive (on X) and $\forall x, y \in X \exists z \in X (xRz \ \& \ yRz)$.) Sobociński [1] defines $K2$ and $K3$ as the normal logics defined by, respectively, $S4.2$ and $\ast 5$, and $S4.3$ and $\ast 5$. Using the name $K1$ for $S4.1$, as Sobociński does, we prove the following result, first established—by a different method—in Lemmon & Scott [1]:

THEOREM 2.4. *$K1$ is determined by the class of all reflexive, transitive McKinsey frames, $K2$ by the class of all directed McKinsey frames, $K3$ by the class of all linear McKinsey frames.*

PROOF. The case of $K1$ is immediate: if a formula A fails in $\mathfrak{M}_{S4.1}$,

then by Lemma 2.3, A fails in a reflexive, transitive McKinsey model. The case of $K2$ is also straightforward. For $K3$, suppose that A fails at x in the canonical model for $K3$. Then, by Lemma 1.2, A fails in $\mathfrak{M}^x_{K3}$. Let Ψ_0 be the set of subformulas of A. Applying Lemma 2.3 we conclude that A fails in a finite, reflexive, transitive McKinsey filtration $\mathfrak{B}$ of $\mathfrak{M}^x_{K3}$. As is easily proved, $\mathfrak{M}^x_{K3}$ is connected, hence $\mathfrak{B}$ is connected too. If $\mathfrak{B}$ is not already anti-symmetric, we proceed as in the proof of Theorem 2.2. So in either case we end up with a linear McKinsey model in which A fails.

COROLLARY 2.5. *K1, K2, K3 are decidable.*

PROOF. The proof of Theorem 2.4 shows that *K1*, *K2*, *K3* all have the finite model property.

The decidability of *K1* was proved in [3]. That *K3* is decidable follows from a general theorem in Bull [2]. For *K2* the result seems to be new.

S4.1 is one of five classical modal logics for which the decision problem was left open in Lemmon & Scott [1]. The four others are *KE*, *KE4*, *DE*, *DE4*, defined as follows. *KE* is the normal logic defined by *K* and the schema

$$\#8.\ \Diamond\Box\mathbf{A} \rightarrow \Box\mathbf{A}.$$

KE4 is the normal logic defined by *KE* and #4. *DE* (*DE4*) is the normal logic defined by *KE* (*KE4*) and the schema

$$\#9.\ \Box\mathbf{A} \rightarrow \Diamond\mathbf{A}.$$

It is proved in [4] that

THEOREM 2.6. *KE, KE4, DE, DE4 are decidable.*

The proofs make use of the completeness results for these logics given in Lemmon & Scott [1], and consist in an analysis of particular filtrations. Although simple, they are too long to be given here.

W is the normal logic defined by *K* and #9 together with the schema

$$\#10.\ \Diamond\mathbf{A} \rightarrow \Box\mathbf{A}.$$

With a different axiomatization, *W* was discussed in von Wright [1]. Using Lemma 1.1 it is possible to prove that *W* is determined

by the class of all frames whose accessibility relations are functions. Indeed, this is precisely what is proved in Lemmon & Scott [1]. However, there is a particularly simple completeness proof for W, published in **[1]**, which has the additional virtue of being constructive. It hinges on the fact that W is strong enough to make the modal operator 'commute' with all the Boolean constants. Let $\mathfrak{N} = \langle N, S \rangle$ be called the standard frame for W, where N is the set of non-negative integers, and, for all m, $n \geq 0$, $S(m, n)$ iff $m + 1 = n$. In effect this transforms clause 4 of the truth definition into 4'. $\|\Box \mathbf{A}\|_n^{\mathfrak{N}} = \|\mathbf{A}\|_{n+1}^{\mathfrak{N}}$. We adopt the convention that $\Box^n$ and $\Diamond^n$ denote, respectively, strings of n occurrences of $\Box$ and $\Diamond$ $(n \geq 0)$.

THEOREM 2.7. *A formula is derivable in W iff it is valid in* $\mathfrak{N}$.

PROOF. If $\mathbf{A}$ is not derivable in W, neither is the formula $\mathbf{A}'$, obtained by letting all the modal operators commute with the Boolean constants until each modal operator is succeeded either by a propositional letter or by another modal operator. Then $\mathbf{A}'$ may be described as a Boolean combination of 'atomic' formulas of type $\Box^m \mathbf{P}_k$. $\mathbf{A}'$ is not a tautology; hence there is an assignment of truth-values T, F to the atomic formulas which assigns F to $\mathbf{A}'$. In an intuitively obvious way this assignment can be extended to a valuation in $\mathfrak{N}$. The formal construction is carried out in **[1]**.

COROLLARY 2.8. *W is decidable.*

This result was obtained before Lemmon & Scott [1] became available. It was also independent of Clifford [1] and Åqvist [1], where other methods are used to derive the same theorems. Another result in **[1]** is

THEOREM 2.9. *W is not finitely axiomatizable.*

PROOF. For positive m, let W_m be the (not necessarily normal) logic defined by K and all the schemata

$$\#(1, n).\ \Box^n \mathbf{A} \leftrightarrow \Diamond^n \mathbf{A},$$
$$\#(2, n).\ \Box^n [\mathbf{A} \to \mathbf{B}] \leftrightarrow [\Box^n \mathbf{A} \to \Box^n \mathbf{B}],$$

for which $1 \leq n \leq m$. One can prove that, for all $m < n$, $W_m \subset W_n \subset$ $\subset W$, and that $W = \bigcup_{m=1}^{\infty} W$. Suppose that W were finitely axiomatizable. Then there would be axiom schemata $S_1, \ldots, S_p$ which, together with K, would yield all of W. We say that an instance of a schema is *minimal* if there is no shorter instance of the same schema. For each i, $1 \leq i \leq p$, there is a positive integer m_i such that some minimal instance of S_i is derivable in W_{m_i}. Then, by the definition of W_{m_i}, all instances of S_i are so derivable. Let $r =$ $= \max\{m_1, \ldots, m_p\}$. It follows that all instances of $S_1, \ldots, S_p$ are derivable in W_r. Consequently $W_r = W$, which contradicts the fact that $W_r \subset W$. Q.E.D.

Although the method of this proof was found independently, the author later became aware that it was used earlier in Kripke [3] and Lemmon [1] to prove that the Feys-von Wright system T is not finitely axiomatizable. Lemmon's paper, which also settles the question of finite axiomatizability for several other logics, contains a reference to a general theorem of Tarski.

III. RESULTS IN THREE-VALUED MODAL LOGIC

A frame $\mathfrak{F}$, in this chapter, is a triple $\langle X, R', R''\rangle$ such that X is a non-empty set (the domain of $\mathfrak{F}$), and R' and R'' are subsets of $X\times X$ (the first and the second accessibility relation of $\mathfrak{F}$). A valuation φ in $\mathfrak{F}$ is a function from the set of propositional letters to the power set of $X\times X$ such that, if φ^1 and φ^2 denote the projections of φ on, respectively, the first and the second coordinate, then, for all $k\geq 0$, $\varphi^1(\mathbf{P}_k)\cap\varphi^2(\mathbf{P}_k)=\emptyset$. A model is a quadruple $\langle X, R', R'', \varphi\rangle$ such that $\langle X, R', R''\rangle$ is a frame and φ is a valuation in that frame.

Throughout this chapter, let $\Phi=\Phi(\bot, \rightarrow, \mathbf{T}, \boxed{1}, \boxed{2})$. The definition of truth at an element x in a model $\mathfrak{A}=\langle X, R', R'', \varphi\rangle$ goes as follows. We assume that T, U, F are three fixed objects.

1. *For all* $k\geq 0$, $\|\mathbf{P}_k\|_x^{\mathfrak{A}}=\begin{cases}\mathrm{T}, \textit{ if } x\in\varphi^1(\mathbf{P}_k),\\ \mathrm{U}, \textit{ if } x\notin\varphi^1(\mathbf{P}_k)\cup\varphi^2(\mathbf{P}_k),\\ \mathrm{F}, \textit{ if } x\in\varphi^2(\mathbf{P}_k).\end{cases}$

2. $\|\bot\|_x^{\mathfrak{A}}=\mathrm{F}$.

3. $\|\mathbf{A}\rightarrow\mathbf{B}\|_x^{\mathfrak{A}}=\begin{cases}\mathrm{T}, \textit{ if } \|\mathbf{A}\|_x^{\mathfrak{A}}\neq\mathrm{U} \textit{ and } \|\mathbf{B}\|_x^{\mathfrak{A}}=\mathrm{T}, \textit{ or}\\ \quad\|\mathbf{A}\|_x^{\mathfrak{A}}=\|\mathbf{B}\|_x^{\mathfrak{A}}=\mathrm{F},\\ \mathrm{U}, \textit{ if } \|\mathbf{A}\|_x^{\mathfrak{A}}=\mathrm{U} \textit{ or } \|\mathbf{B}\|_x^{\mathfrak{A}}=\mathrm{U},\\ \mathrm{F}, \textit{ if } \|\mathbf{A}\|_x^{\mathfrak{A}}=\mathrm{T} \textit{ and } \|\mathbf{B}\|_x^{\mathfrak{A}}=\mathrm{F}.\end{cases}$

4. $\|\mathbf{TA}\|_x^{\mathfrak{A}}=\begin{cases}\mathrm{T}, \textit{ if } \|\mathbf{A}\|_x^{\mathfrak{A}}=\mathrm{T},\\ \mathrm{F}, \textit{ if } \|\mathbf{A}\|_x^{\mathfrak{A}}\neq\mathrm{T}.\end{cases}$

$$5.\ \|\boxed{1}\mathbf{A}\|_x^{\mathfrak{A}} = \begin{cases} \mathrm{T}, \textit{ if } \|\mathbf{A}\|_x^{\mathfrak{A}} \neq \mathrm{U}, \textit{ and, for all } y \textit{ such that} \\ \quad xR'y,\ \|\mathbf{A}\|_y^{\mathfrak{A}} = \mathrm{T}, \\ \mathrm{U}, \textit{ if } \|\mathbf{A}\|_x^{\mathfrak{A}} = \mathrm{U}, \\ \mathrm{F}, \textit{ if } \|\mathbf{A}\|_x^{\mathfrak{A}} \neq \mathrm{U}, \textit{ and, for some } y \textit{ such} \\ \quad \textit{that } xR'y,\ \|\mathbf{A}\|_y^{\mathfrak{A}} \neq \mathrm{T}. \end{cases}$$

$$6.\ \|\boxed{2}\mathbf{A}\|_x^{\mathfrak{A}} = \begin{cases} \mathrm{T}, \textit{ if } \|\mathbf{A}\|_x^{\mathfrak{A}} \neq \mathrm{U}, \textit{ and, for all } y \textit{ such that} \\ \quad xR''y,\ \|\mathbf{A}\|_y^{\mathfrak{A}} \neq \mathrm{F}, \\ \mathrm{U}, \textit{ if } \|\mathbf{A}\|_x^{\mathfrak{A}} = \mathrm{U}, \\ \mathrm{F}, \textit{ if } \|\mathbf{A}\|_x^{\mathfrak{A}} \neq \mathrm{U}, \textit{ and, for some } y \textit{ such} \\ \quad \textit{that } xR''y,\ \|\mathbf{A}\|_y^{\mathfrak{A}} = \mathrm{F}. \end{cases}$$

A is said to be acceptable in $\mathfrak{A}$ iff, for all x in $\mathfrak{A}$, $\|\mathbf{A}\|_x^{\mathfrak{A}} \neq \mathrm{F}$. A formula is valid in a frame iff it is acceptable in all models on that frame. A formula is valid in a class of frames iff it is valid in each of the frames.

Let H_0 denote the set of formulas true in every model by virtue of the first four clauses alone. Note that $PC \subseteq H_0$. If A is a formula, let $\Omega(\mathbf{A})$ denote the set of propositional letters that have at least one occurrence in A which is not within the scope of any T. The rule of modus ponens appears, in this chapter only, in a weakened form:

$$\text{WMP.}\quad \frac{\mathbf{A},\ \mathbf{A} \to \mathbf{B}}{\mathbf{B}},\ \text{if } \Omega(\mathbf{A}) \subseteq \Omega(\mathbf{B}).$$

A new rule is that of acceptability:

$$\text{Acc.}\quad \frac{\mathbf{A}}{\neg\mathrm{T}\neg\mathbf{A}.}$$

Although H_0 is but one of many three-valued systems, we bluntly say that L is a *three-valued modal logic* iff $H_0 \subseteq L \subseteq \Phi$, and L is closed under weakened modus ponens and acceptability. Note that this makes H_0 the smallest three-valued modal logic. We define the following necessitation rules:

RN1. $\dfrac{A \to B}{\boxed{1}A \to \boxed{1}B}$, if $\Omega(B) \subseteq \Omega(A)$.

RN2. $\dfrac{A}{\boxed{2}A}$.

A three-valued modal logic is said to be *normal* if it is closed under RN1 and RN2 and contains every instance of the following schemata:

*0. $\boxed{1}[TA \to TA]$.
*1. $\boxed{1}A \leftrightarrow \boxed{1}TA$.
*2. $\boxed{2}A \leftrightarrow \boxed{2}\neg T\neg A$.
*3. $+A \leftrightarrow +\boxed{1}A$.
*4. $+A \leftrightarrow +\boxed{2}A$.
*5. $\boxed{1}[A \to B] \to [\boxed{1}A \to \boxed{1}B]$.
*6. $\boxed{2}[A \to B] \to [\boxed{2}A \to \boxed{2}B]$, where $\Omega(A) \subseteq \Omega(B)$.

The smallest normal logic is denoted by *H*. *H* is a three-valued, bi-modal counterpart of the classical modal logic *K* of Chapter I, and the completeness proof for *H* parallels that for *K*. Let *L* be a three-valued modal logic. A set $\Sigma \subseteq \Phi$ is said to be *L*-full iff

(i) for all **A**, if $\vdash_L \mathbf{A}$ then $\neg\mathbf{A} \notin \Sigma$;
(ii) Σ is *L*-consistent;
(iii) for all **A** and **B**, $\mathbf{A} \vee \mathbf{B} \in \Sigma$ iff $\mathbf{A},\mathbf{B} \in \Sigma$ or $\neg\mathbf{A},\mathbf{B} \in \Sigma$ or $\neg\mathbf{A},\neg\mathbf{B} \in \Sigma$; $\neg[\mathbf{A} \to \mathbf{B}] \in \Sigma$ iff $\mathbf{A},\neg\mathbf{B} \in \Sigma$;
(iv) for all **A**, $T\mathbf{A} \in \Sigma$ iff $\mathbf{A} \in \Sigma$; $\neg T\mathbf{A} \in \Sigma$ iff $\mathbf{A} \notin \Sigma$;
(v) for all **A**, $\boxed{1}\mathbf{A} \in \Sigma$ or $\neg\boxed{1}\mathbf{A} \in \Sigma$ iff $\mathbf{A} \in \Sigma$ or $\neg\mathbf{A} \in \Sigma$;
(vi) for all **A**, $\boxed{2}\mathbf{A} \in \Sigma$ or $\neg\boxed{2}\mathbf{A} \in \Sigma$ iff $\mathbf{A} \in \Sigma$ or $\neg\mathbf{A} \in \Sigma$.

By the *canonical model* (for *L*) we mean $\mathfrak{M}_L = \langle X_L, R'_L, R''_L, \varphi_L \rangle$ where

$X_L = \{x : x \subseteq \Phi \ \&\ x \text{ is } L\text{-full}\}$;
$R'_L = \{\langle x, y\rangle : x,y \in X_L \ \&\ \forall\mathbf{A}(\boxed{1}\mathbf{A} \in x \Rightarrow \mathbf{A} \in y)\}$;
$R''_L = \{\langle x, y\rangle : x,y \in X_L \ \&\ \forall\mathbf{A}(\boxed{2}\mathbf{A} \in x \Rightarrow \neg\mathbf{A} \notin y)\}$;
$\varphi^1_L(\mathbf{P}_k) = \{x : x \in X_L \ \&\ \mathbf{P}_k \in x\}$, for all $k \geq 0$,
$\varphi^2_L(\mathbf{P}_k) = \{x : x \in X_L \ \&\ \neg\mathbf{P}_k \in x\}$, for all $k \geq 0$.

Evidently $\mathfrak{M}_L$ is a model.

LEMMA 3.1. *If L is a three-valued modal logic, then for all x in $\mathfrak{M}_L$ and for all formulas* A,

$$\|\mathbf{A}\|_x^{\mathfrak{M}_L} = \begin{cases} \mathrm{T}, \text{ if } \mathbf{A} \in x, \\ \mathrm{U}, \text{ if } \mathbf{A} \notin x \text{ and } \neg\mathbf{A} \notin x, \\ \mathrm{F}, \text{ if } \neg\mathbf{A} \in x. \end{cases}$$

THEOREM 3.2. *H is determined by the class of all frames.*

Consider the following schemata:

*7. $\boxed{1}\mathbf{A} \to \boxed{2}\mathbf{A}$.
*8. $\boxed{1}[\mathbf{A} \to \mathbf{A}] \to [\boxed{2}\mathbf{A} \to \boxed{1}\mathbf{A}]$.
*9. $\boxed{1}\mathbf{A} \to \mathbf{A}$.
*10. $\boxed{2}\mathbf{A} \to \mathbf{A}$.
*11. $\boxed{1}\mathbf{A} \to \boxed{1}\boxed{1}\mathbf{A}$.
*12. $\boxed{2}\mathbf{A} \to \boxed{2}\boxed{2}\mathbf{A}$.
*13. $\boxed{1}[\mathbf{A} \to \mathbf{A}] \to [\neg\boxed{1}\neg\boxed{1}\mathbf{A} \to \mathbf{A}]$.
*14. $\neg\boxed{2}\neg\boxed{2}\mathbf{A} \to \mathbf{A}$.

For $7 \leq n \leq 14$, let H^*n be the normal logic defined by H and $^*n$.

THEOREM 3.3. *For $7 \leq n \leq 14$, H^*n is determined by the class of all frames $\mathfrak{F} = \langle X, R', R'' \rangle$ that satisfy the following conditions*: (7) *$R'' \subseteq R'$*; (8) *$R' \subseteq R''$*; (9) *R' is reflexive*; (10) *R'' is reflexive*; (11) *R' is transitive*; (12) *R'' is transitive*; (13) *R' is symmetric*; (14) *R'' is symmetric*.

Moreover, these results are additive in the sense that if several of the schemata are added to H, then the ensuing logic is determined by the class of all frames satisfying all the corresponding conditions. Some of these cases are singled out in [2], where it is also proved that H and all extensions of H in terms of $^{**}7$—14 are decidable.

The axiomatizations used here differ slightly from those used in [2], but it is easy to verify that two versions are equivalent. One feature of the present choice of schemata is that T does not occur in $^{**}7$—14. It is possible to prove, among other things, that Prior's system Q, first defined in Prior [1], is axiomatized by a set of

axioms for *PC* together with **5—14 and the rules MP, RN1, and RN2; a proof of this contention will be published elsewhere. This axiomatization has the advantage over those previously known, Bull [1] and Prior [2], that it lays bare the similarity between *Q* and the classical modal logic *S5*.

IV. RESULTS CONCERNING EXTENSIONS OF MINIMAL LOGIC

In this chapter, a frame $\mathfrak{F}$ is a triple $\langle X, R, Q\rangle$, where X is a non-empty set (the domain of $\mathfrak{F}$), R is a partial ordering of X (the accessibility relation of $\mathfrak{F}$), and Q is an R-hereditary subset of X. (A set Y is R-hereditary iff $\forall x,y(xRy \;\&\; x\in Y \Rightarrow y\in Y)$.) If $Q=\emptyset$, $\mathfrak{F}$ is said to be normal. A valuation in $\mathfrak{F}$ is a function assigning to each propositional letter an R-hereditary subset of X. $\mathfrak{A}=\langle X, R, Q, \varphi\rangle$ is a model iff $\langle X, R, Q\rangle$ is a frame, and φ a valuation in that frame.

The set of formulas, in this chapter, is $\Phi=\Phi(\bot, \wedge, \vee, \rightarrow)$. The definition of truth at a point x in a model $\mathfrak{A}=\langle X, R, Q, \varphi\rangle$ is as follows. Assume that T and F are two fixed objects.

1. *For all* $k\geq 0$, $\|\mathbf{P}_k\|_x^{\mathfrak{A}}=\begin{cases}\mathrm{T}, \textit{ if } x\in\varphi(\mathbf{P}_k),\\ \mathrm{F}, \textit{ if } x\notin\varphi(\mathbf{P}_k).\end{cases}$

2. $\|\bot\|_x^{\mathfrak{A}}=\begin{cases}\mathrm{T}, \textit{ if } x\in Q,\\ \mathrm{F}, \textit{ if } x\notin Q.\end{cases}$

3. $\|\mathbf{A}\wedge\mathbf{B}\|_x^{\mathfrak{A}}=\begin{cases}\mathrm{T}, \textit{ if } \|\mathbf{A}\|_x^{\mathfrak{A}}=\|\mathbf{B}\|_x^{\mathfrak{A}}=\mathrm{T},\\ \mathrm{F}, \textit{ if } \|\mathbf{A}\|_x^{\mathfrak{A}}=\mathrm{F} \textit{ or } \|\mathbf{B}\|_x^{\mathfrak{A}}=\mathrm{F}.\end{cases}$

4. $\|\mathbf{A}\vee\mathbf{B}\|_x^{\mathfrak{A}}=\begin{cases}\mathrm{T}, \textit{ if } \|\mathbf{A}\|_x^{\mathfrak{A}}=\mathrm{T} \textit{ or } \|\mathbf{B}\|_x^{\mathfrak{A}}=\mathrm{T},\\ \mathrm{F}, \textit{ if } \|\mathbf{A}\|_x^{\mathfrak{A}}=\|\mathbf{B}\|_x^{\mathfrak{A}}=\mathrm{F}.\end{cases}$

5. $\|\mathbf{A}\rightarrow\mathbf{B}\|_x^{\mathfrak{A}}=\begin{cases}\mathrm{T}, \textit{ if, for all } y \textit{ such that } xRy, \textit{ if}\\ \quad \|\mathbf{A}\|_y^{\mathfrak{A}}=\mathrm{T} \textit{ then } \|\mathbf{B}\|_y^{\mathfrak{A}}=\mathrm{T},\\ \mathrm{F}, \textit{ if, for some } y \textit{ such that } xRy,\\ \quad \|\mathbf{A}\|_y^{\mathfrak{A}}=\mathrm{T} \textit{ and } \|\mathbf{B}\|_y^{\mathfrak{A}}=\mathrm{F}.\end{cases}$

A is true in $\mathfrak{A}$ iff, for all x in $\mathfrak{A}$, $\|\mathbf{A}\|_x^{\mathfrak{A}} = \mathrm{T}$. A formula is valid in a frame iff it is true in all models on it. A formula is valid in a class of frames iff it is valid in each frame.

The logics treated in this chapter all contain the so-called minimal logic, first defined in Johansson [1]. One standard axiomatization of *J*, as Johansson's logic is called here, is given in **[5]**, but our analysis is independent of any particular axiom system. A completeness theorem for *J* is obtained by suitably modifying the argument outlined for the classical modal logic *K* in Chapter I. If *L* is a logic and $\Sigma \subseteq \Phi$, then Σ is said to be *L*-full iff

(i) $L \subseteq \Sigma$;
(ii) Σ is *L*-consistent;
(iii) for all **A** and **B**, $\mathbf{A} \wedge \mathbf{B} \in \Sigma$ iff $\mathbf{A} \in \Sigma$ and $\mathbf{B} \in \Sigma$;
(iv) for all **A** and **B**, $\mathbf{A} \vee \mathbf{B} \in \Sigma$ iff $\mathbf{A} \in \Sigma$ or $\mathbf{B} \in \Sigma$;
(v) for all **A** and **B**, if $\mathbf{A} \to \mathbf{B} \in \Sigma$ and $\mathbf{A} \in \Sigma$ then $\mathbf{B} \in \Sigma$.

By the *canonical model* (for *L*) we mean $\mathfrak{M}_L = \langle X_L, \subseteq, Q_L, \varphi_L \rangle$ where

$X_L = \{x : x \subseteq \Phi \ \&\ x \text{ is } L\text{-full}\}$;
$Q_L = \{x : x \in X_L \ \&\ \bot \in x\}$;
$\varphi_L(\mathbf{P}_k) = \{x : x \in X_L \ \&\ \mathbf{P}_k \in x\}$, for all $k \geq 0$.

It is easy to see that $\mathfrak{M}_L$ is indeed a model. Our basic result this time takes the following shape:

LEMMA 4.1. *If L is a logic containing J, then for all x in* $\mathfrak{M}_L$ *and all formulas* **A**, $\|\mathbf{A}\|_x^{\mathfrak{M}_L} = \mathrm{T}$ *iff* $\mathbf{A} \in x$.

This at once gives us:

THEOREM 4.2. *J is determined by the class of all frames.*

In **[5]** the following schemata are considered:

I. $\bot \to \mathbf{A}$.
Q. $\bot$.
K. $\neg\mathbf{A} \vee \neg\neg\mathbf{A}$.
L. $[\mathbf{A} \to \mathbf{B}] \vee [\mathbf{B} \to \mathbf{A}]$.
L^{N}. $[\mathbf{A} \to \mathbf{B} \vee \bot] \vee [\mathbf{B} \to \mathbf{A} \vee \bot]$.
$\mathrm{L}_1^{\mathrm{Q}}$. $\bot \to [\mathbf{A} \to \mathbf{B}] \vee [\mathbf{B} \to \mathbf{A}]$.

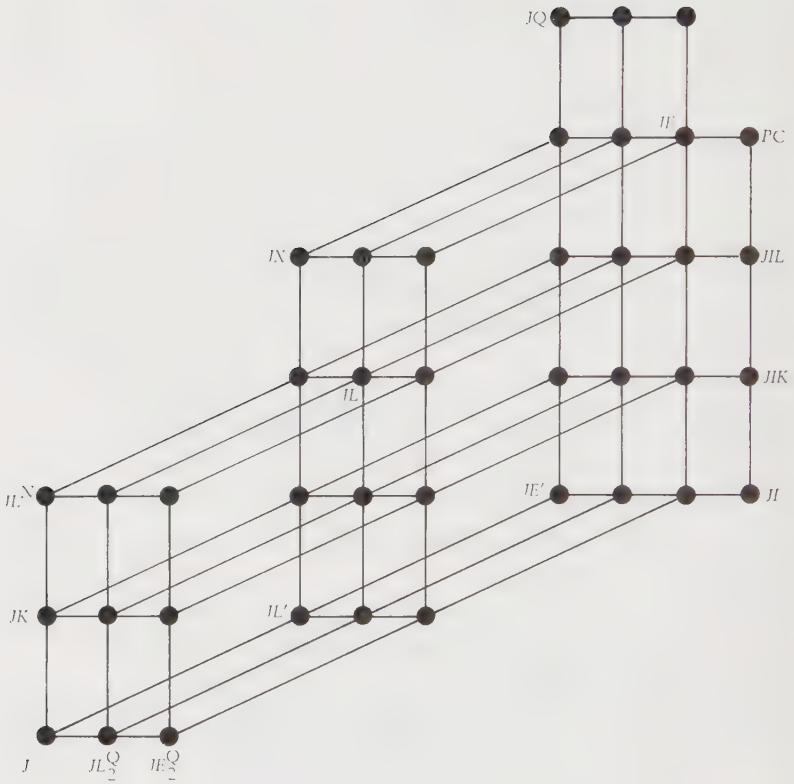
L_2^Q. $[\bot \to [A \to B]] \vee [\bot \to [B \to A]]$.
L'. $\neg A \vee [\bot \to A]$.
E. $A \vee [A \to B]$.
X. $A \vee \neg A$.
E_1^Q. $\bot \to A \vee [A \to B]$.
E_2^Q. $[\bot \to A] \vee [\bot \to [A \to B]]$.
E'. $\bot \vee [\bot \to A]$.

Let the logic generated by J and schemata $S_1, \ldots, S_n$ be denoted by $JS_1 \ldots S_n$. A binary relation R defined on a set X satisfies a certain condition on $Y \subseteq X$ if the condition is satisfied by $R \cap (Y \times Y)$.

THEOREM 4.3. *The following logics are determined by the class of all frames* $\mathfrak{F} = \langle X, R, Q \rangle$ *satisfying the following conditions*:

JI: $Q = \emptyset$.
JQ: $Q = X$.
JK: *R is directed on X—Q.*
JL: *R is linear.*
JL^N: *R is linear on X—Q.*
JL_1^Q: $\forall x,y,z(xRy \ \& \ xRz \ \& \ x \in Q \Rightarrow yRz \vee zRy)$.
JL_2^Q: *R is linear on Q.*
JL': $\forall x,y(x \in X—Q \ \& \ y \in Q \Rightarrow xRy)$.
JE: *R is symmetric (that is, R becomes the identity).*
JX: *R is symmetric on X—Q.*
JE_1^Q: *R is symmetric on Q.*
JE_2^Q: *R is universal on Q.*
JE': $Q = \emptyset$ *or* $Q = X$.

The above results are additive. Thus, for example, as was proved in Dummett & Lemmon [1], *JIK* and *JIL* are determined, respectively, by the class of normal directed and normal linear frames. The relationship between the logics definable in terms of the above schemata, excepting L_1^Q and E_1^Q, appears in the chart below. (All logics represented are distinct. A logic L_1 is contained in a logic L_2 iff there is a path leading from L_1 to L_2 which at every point is either rising or horizontal and directed to the right.) *JI* is identical with Heyting's propositional logic first defined in Heyting [1]. *JIK* and *JIL* were studied, independently of



Dummett and Lemmon, in Umezawa [1], [2], where infinitely many logics between *JI* and *PC* are considered, including the family JIF_m mentioned below. *JX* and *JE*, with different axiomatizations, are treated in Curry [1] and Kanger [1], respectively.

Consider the following family of schemata ($m \geq 1$):

$$F_m.\ \mathbf{A}_1 \vee [\mathbf{A}_1 \to \mathbf{A}_2] \vee \ldots \vee [\mathbf{A}_m \to \mathbf{A}_{m+1}].$$

THEOREM 4.4. *JF_m is determined by the class of all linear frames containing m elements.*

Thus we obtain the result in Thomas [1] that JIF_m is determined by the class of all normal linear frames with m elements.

It is proved in [5] that all logics definable by J and schemata mentioned here, other than L_1^Q and E_1^Q, are decidable. In the case of the logics $JF_m(JIF_m, JQF_m)$ this is obvious since, by Theorem 4.4, they are finitely many-valued logics.

REFERENCES

JSL = Journal of Symbolic Logic.
NDJFL = Notre Dame Journal of Formal Logic.
ZMLGM = Zeitschrift für mathematische Logik und Grundlagen der Mathematik.

Lennart Åqvist, [1] "'Next' and 'ought'. Alternative foundations for von Wright's tense-logic, with an application to deontic logic", *Logique et analyse*, 9, 1966, pp. 231—251.

R. A. Bull, [1] "An axiomatization of Prior's modal calculus Q", *NDJFL*, 5, 1964, pp. 211—214. [2] "That all normal extensions of S4.3 have the finite model property", *ZMLGM*, 12, 1966, pp. 341—344.

John E. Clifford, [1] "Tense logic and the logic of change", *Logique et analyse*, 9, 1966, pp. 219—230.

Haskell B. Curry, [1] *A theory of formal deducibility*, Notre Dame 1950.

M. A. E. Dummett & E. J. Lemmon, [1] "Modal logics between S4 and S5", *ZMLGM*, 5, 1959, pp. 250—264.

Arend Heyting, [1] "Die formalen Regeln der intuitionistischer Logik", *Sitzungsberichte der preussischen Akademie der Wissenschaften, physikalisch-mathematische Klasse*, Berlin 1930, pp. 42—56.

Jaakko Hintikka, [1] "Modality and quantification", *Theoria*, 27, 1961, pp. 119—128. [2] "The modes of modality", *Acta Philosophica Fennica*, fasc. 16, 1963, pp. 65—81.

Ingebrigt Johansson, [1] "Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus", *Compositio mathematica*, 4, 1937, pp. 119—136.

Stig Kanger, [1] "A note on partial postulate sets for propositional logic", *Theoria*, 21, 1955, pp. 99—104. [2] *Provability in logic*, Stockholm 1957. [3] "On the characterization of modalities", *Theoria*, 23, 1957, pp. 152—155.

David Kaplan, [1] Review of Kripke [2], *JSL*, 31, 1966, pp. 120—122.

Saul A. Kripke, [1] "A completeness theorem in modal logic", *JSL*, 24, 1959, pp. 1—14. [2] "Semantical analysis of modal logic I. Normal modal propositional calculi", *ZMLGM*, 9, 1963, pp. 67—96. [3] "Semantical analysis of modal logic II. Non-normal modal propositional calculi", in *The theory of models* (eds. Addison, Henkin & Tarski), Amsterdam 1965, pp. 206—220. [4] "Semantical analysis of intuitionistic logic I", in *Formal systems and recursive functions* (eds. Crossley & Dummett), Amsterdam 1965, pp. 92—130.

E. J. Lemmon, [1] "Some results on finite axiomatizability in modal logic", *NDJFL*, 6, 1965, pp. 301—308.

E. J. Lemmon & Dana Scott, [1] *Intensional logic*, preliminary draft of initial chapters by E. J. Lemmon, July 1966 (mimeographed).

D. Makinson, [1] "On some completeness theorems in modal logic", *ZMLGM*, '12, 1966, pp. 379—384.

J. C. C. McKinsey, [1] "On the syntactical construction of systems of modal logic", *JSL*, 10, 1945, pp. 83—96.

Richard Montague, [1] "Logical necessity, physical necessity, ethics, and quantifiers", *Inquiry*, 3, 1960, pp. 259—269.

Arthur N. Prior, [1] *Time and modality*, Oxford 1957. [2] "Axiomatisations of the modal calculus Q", *NDJFL*, 5, 1964, pp. 215—217.

Krister Segerberg, [1] "On the logic of 'to-morrow'", *Theoria*, 33, 1967, pp. 45—52. [2] "Some modal logics based on a three-valued logic", *Theoria*, 33, 1967, pp. 53—71. [3] "Decidability of S4.1", *Theoria*, 34, 1968, pp. 7—20. [4] "Decidability of four modal logics", *Theoria*, 34, 1968, pp. 21—25. [5] "Propositional logics related to Heyting's and Johansson's", *Theoria*, 34, 1968, pp. 26—61.

Bolesław Sobociński, [1] "Remarks about axiomatizations of certain modal systems", *NDJFL*, 5, 1964, pp. 71—80.

Ivo Thomas, [1] "Finite limitations on Dummett's LC", *NDJFL*, 3, 1962, pp. 170—174.

Toshio Umezawa, [1] "Über die Zwischensysteme der Aussagenlogik", *Nagoya Mathematical Journal*, 9, 1955, pp. 181—189. [2] "On intermediate propositional logics", *JSL*, 24, 1959, pp. 20—36.

G. H. von Wright, [1] "'And next'", *Acta Philosophica Fennica*, fasc. 18, 1965, pp. 293—304.